Advanced Computer Graphics
Tone Mapping / Tone Reproduction

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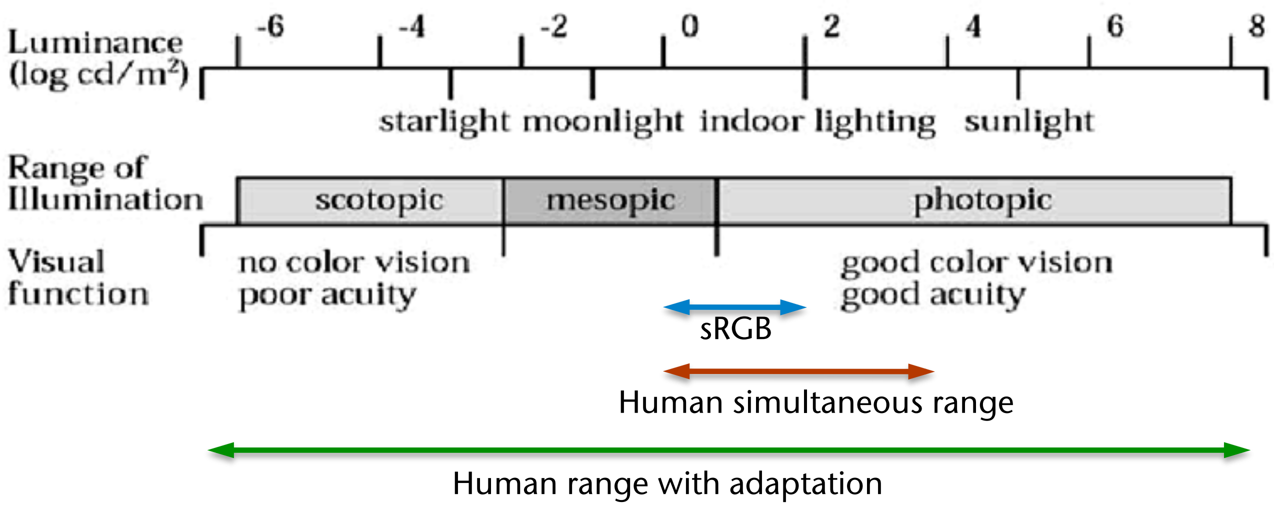
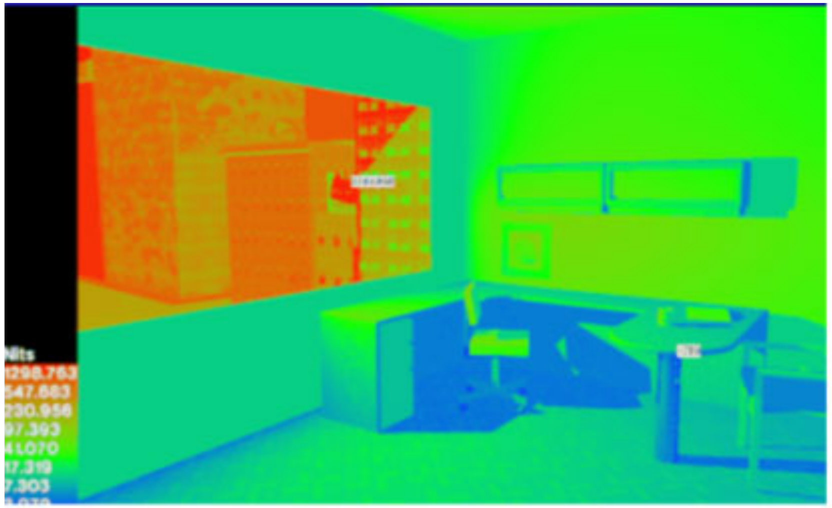
Motivation

- Definition:
  - The dynamic range of an image is the contrast ratio between the brightest and darkest parts
  - The dynamic range of a display or optical sensor is the ratio of the brightest representable or perceived luminance to the darkest
- The dynamic range of the human visual system:



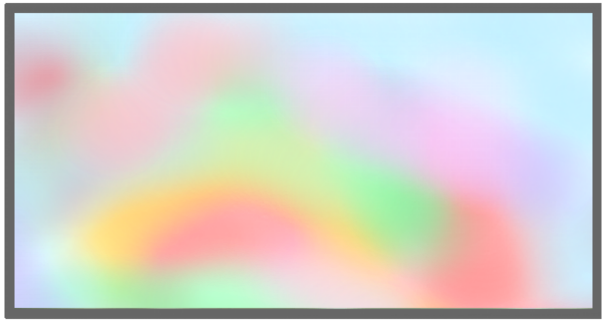
Sources of High Dynamic Range Images (HDRI)

- Ray-Tracing: physically accurate synthetic images
- Photography:
  - Several shots with different exposure times
  - "Blending" together (needs calibrated response curve from camera)

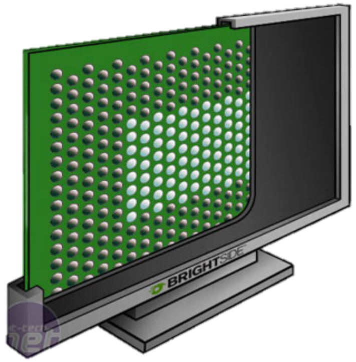


Display of HDR Images

- Use either real HDR displays …



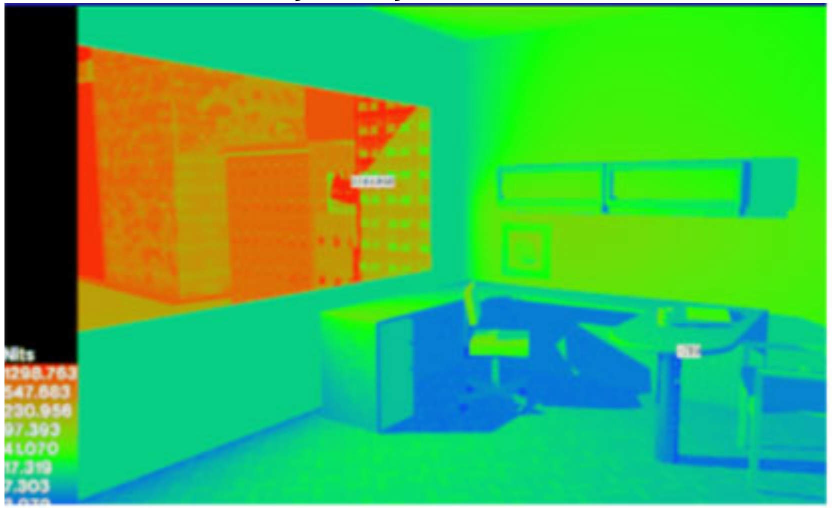
Background illumination of HDR display



- … or LDR displays; then you need:
- Tone mapping (TM) / tone reproduction = Map of the real potential "high dynamic range" (HDR) luminances on a "low dynamic range" (LDR) displays with a limited luminance bandwidth.

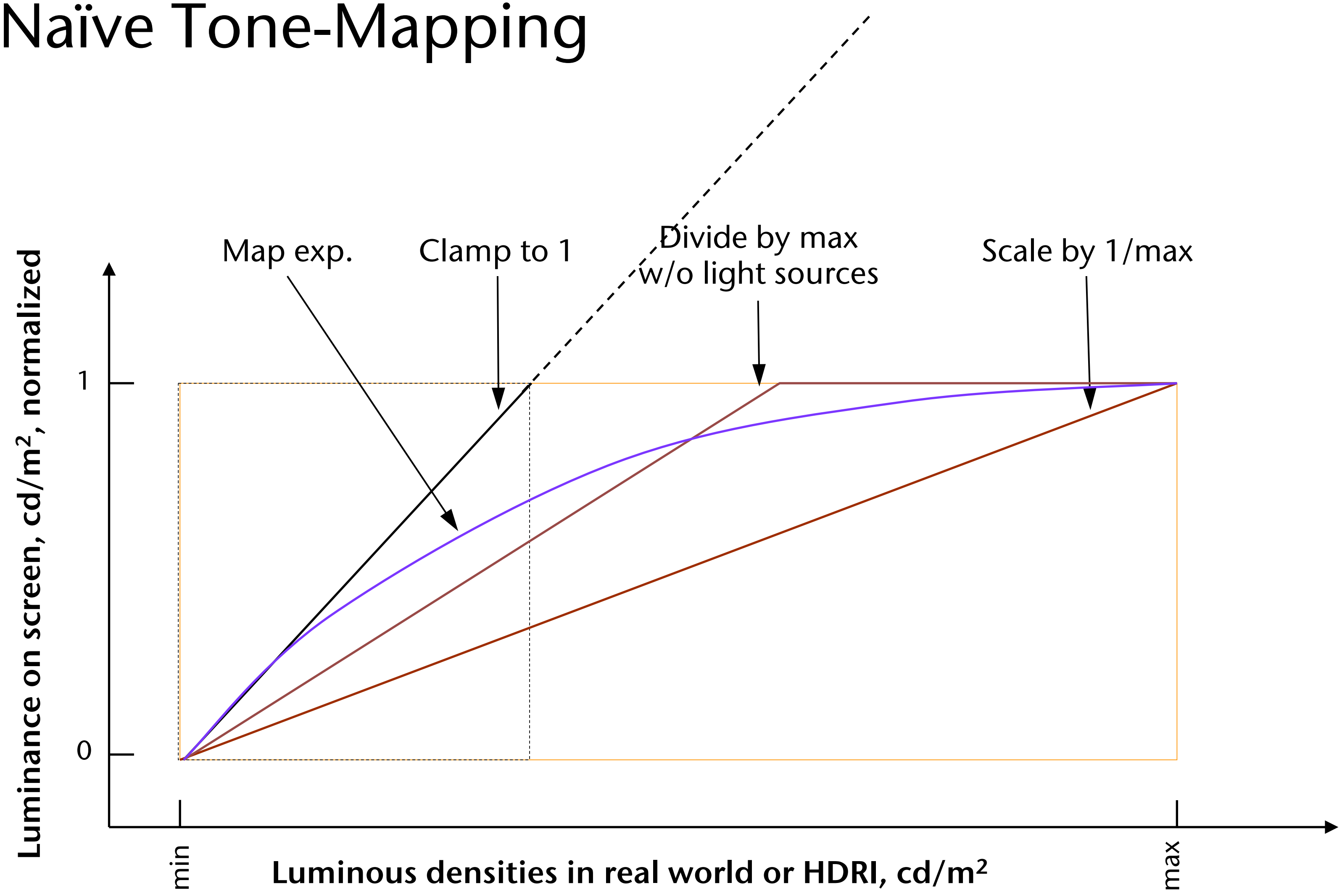
Informal Statement of the Problem

Physically correct



Best effort rendering on LDR display

Naïve Tone-Mapping



Result of the Naive Mapping

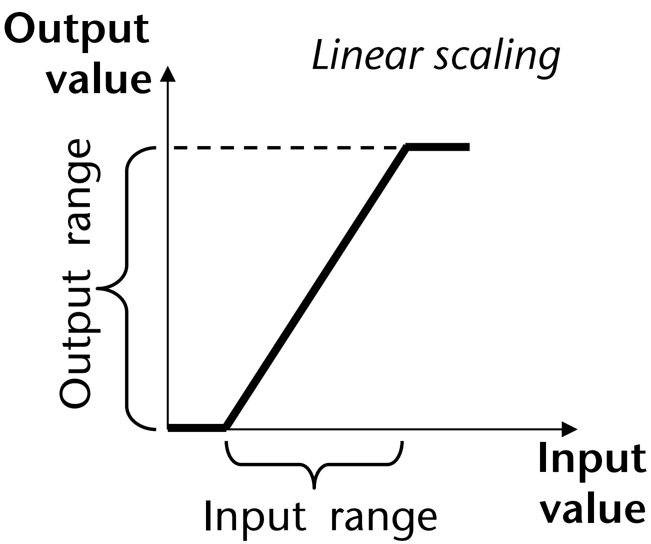
Scale by 1/max

Clamp to 1

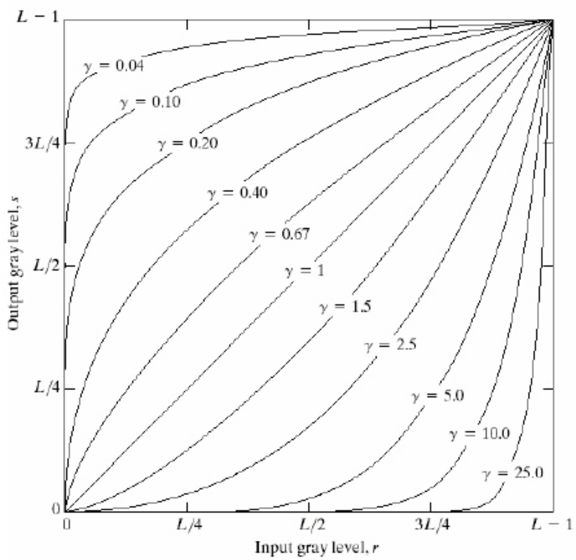
Exp. mapping

An Important Class of Tone Mappings

- First consider pure „point functions":
  - Determine a transfer function $y = T(x)$
    - Also called tone mapping operator
  - T only depends on the pixel position x and its color; it is completely independent of the neighborhood around x
- Examples:

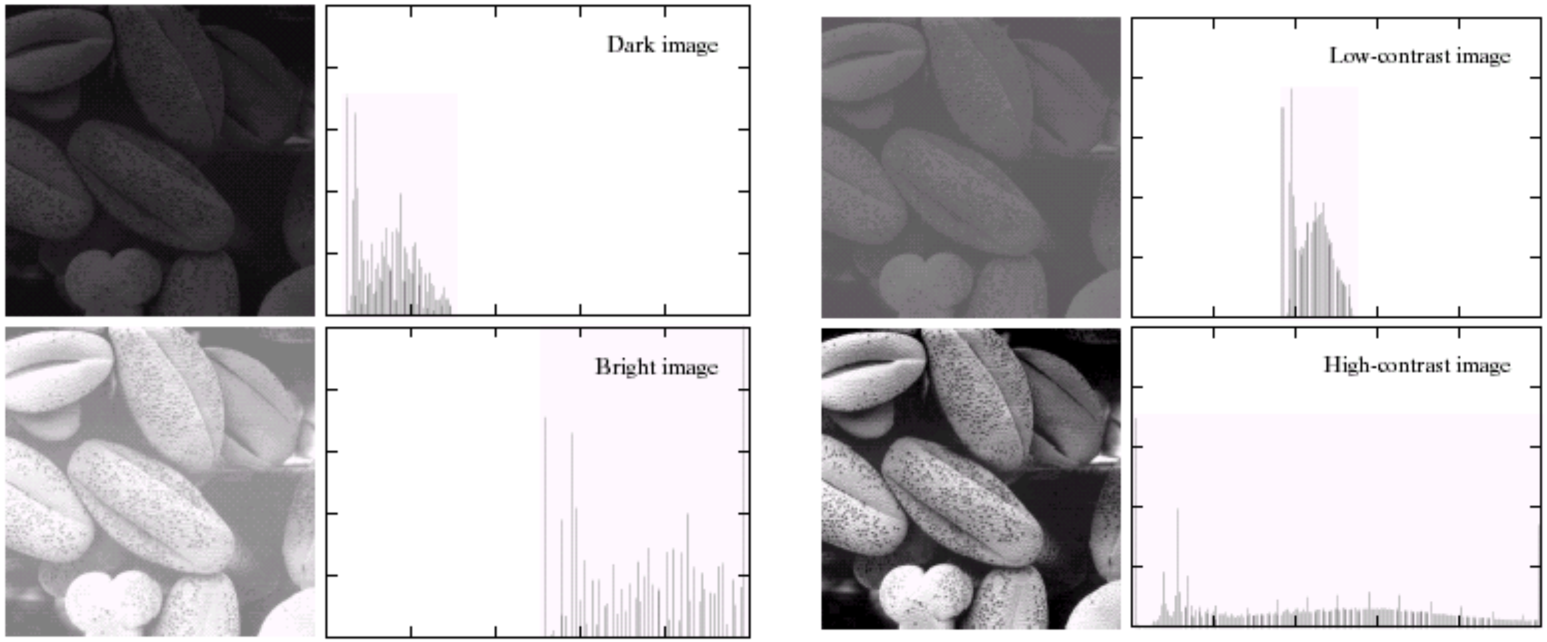


Gamma correction



The Luminance Histogram

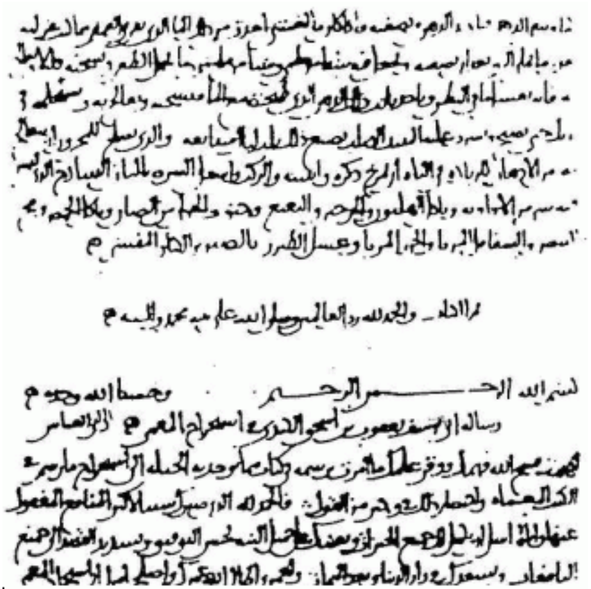
- Images with "unbalanced" histograms do not use the full dynamic range
- Balanced histograms results in a more pleasant image and reflects the content much better



- The histogram of an image contains valuable information about the grayscale
- It contains no spatial information
- All of the following images have exactly the same histogram!

Historical Note: Histograms for Decrypting

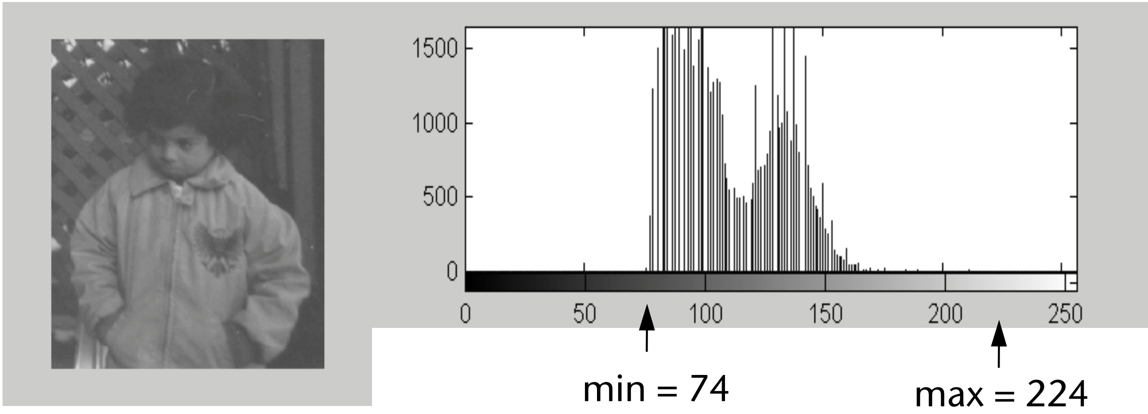
- First presented by Abu Yusuf Ya'qub ibn Ishaq al-Sabbah Al-Kindi as a tool for deciphering a (simple) substitution cipher
  - Now called frequency analysis method
  - Breakthrough at this time, 850 n. Chr. [Simon Singh: The Code Book, 1999]



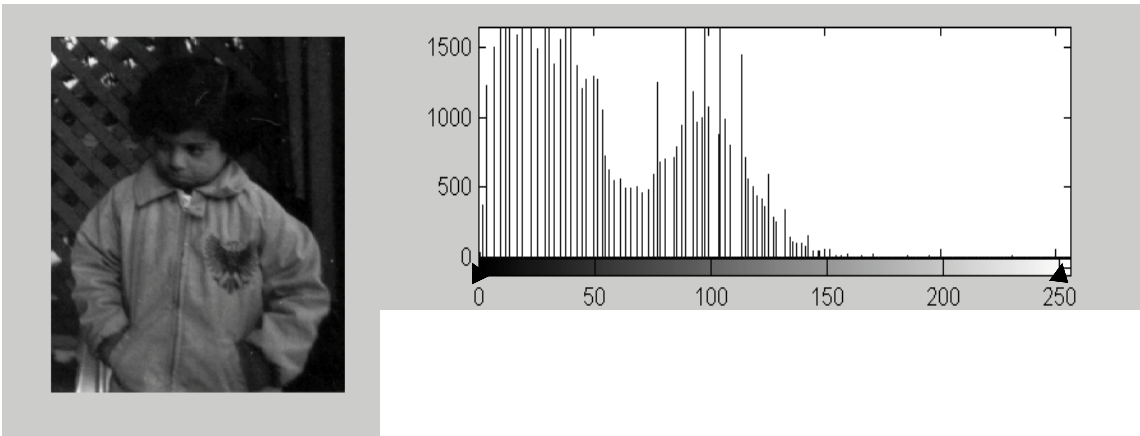
Histogram Stretching

- Linear scaling = "*histogram stretching*": $J = \dfrac{I - I_{\text{min}}}{I_{\text{max}} - I_{\text{min}}} \cdot J_{\text{max}}$

I



J



Interpretation of an Image Histogram

- Treat all pixels as i.i.d. random variables , i.e., each pixel = one RV
  - *i.i.d. random variables = independent, identically distributed RVs*
- Histogram = discrete approximation of the *probability density function (PDF*) of a pixel in the image

Discrete (Histogram) vs. Continuous Formulation (PDF/CDF)

Discrete world:

$x \in 0, \ldots, L-1$

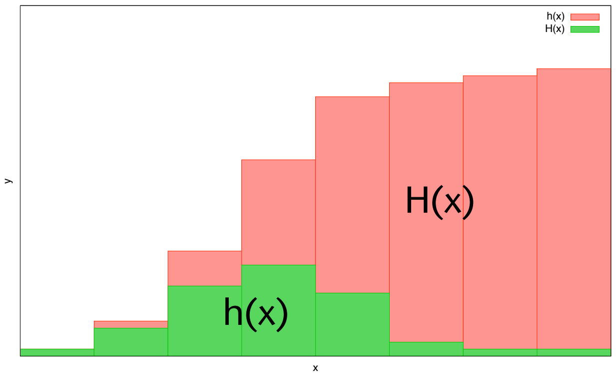
$L = \#$ levels

Histogram:

$h(x) = \#$ pixels with level x

Cumulative histogram:

$$H(x) = \sum_{u=0}^{x} h(u)$$



Continuous world:

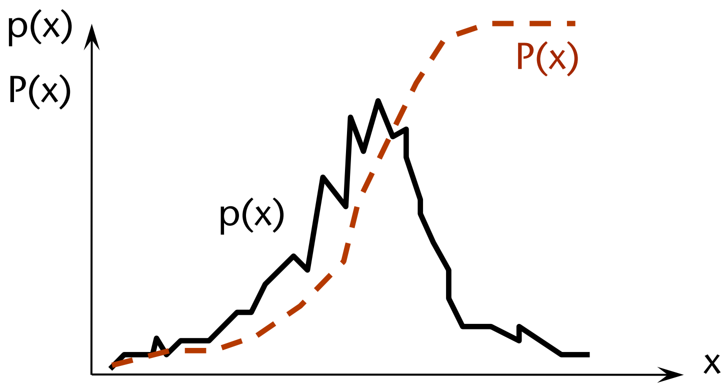
$x \in [0, 1]$

Probability distrib. funct. (PDF):

$p(x) =$ "density" at level x

Cumul. distrib. function (CDF):

$$P(x) = \int_0^x p(u)du$$



- Clearly:

$$H(L-1) = \sum_{u=0}^{L-1} h(u) = N = \text{number of pixels}$$

  - Therefore $h(x)$ respectively $H(x)$ is often normalized with $\frac{1}{N}$
- Let X be a random variable;
  the probability that the event "$X \leq x$" occurs is

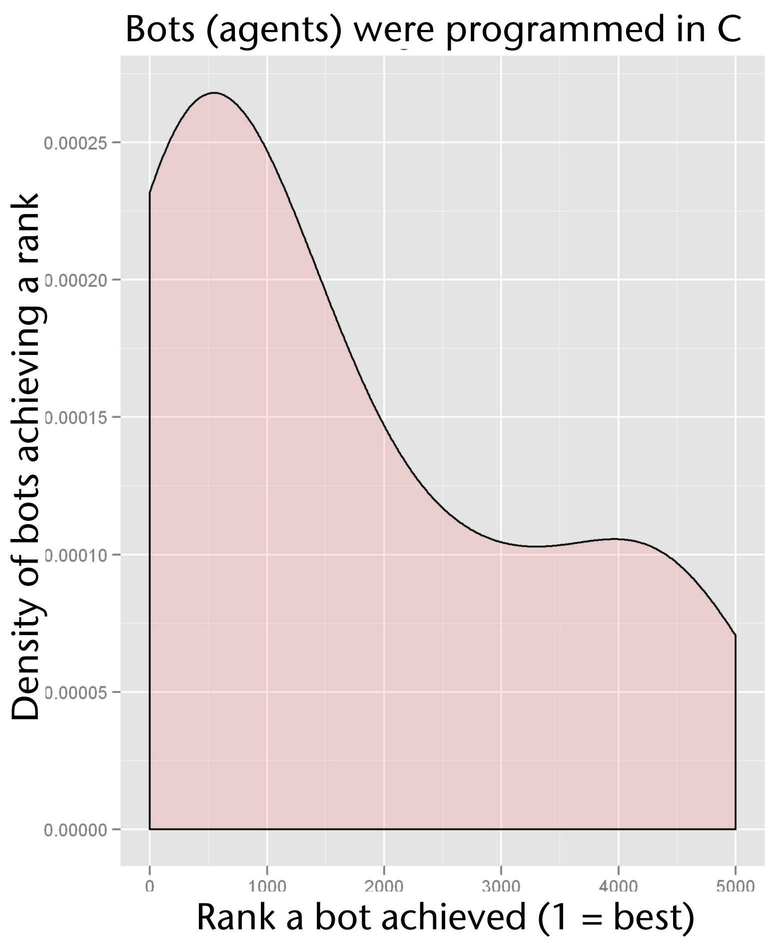
$$P[X \leq x] = P(x) = \int_0^x p(u)du$$

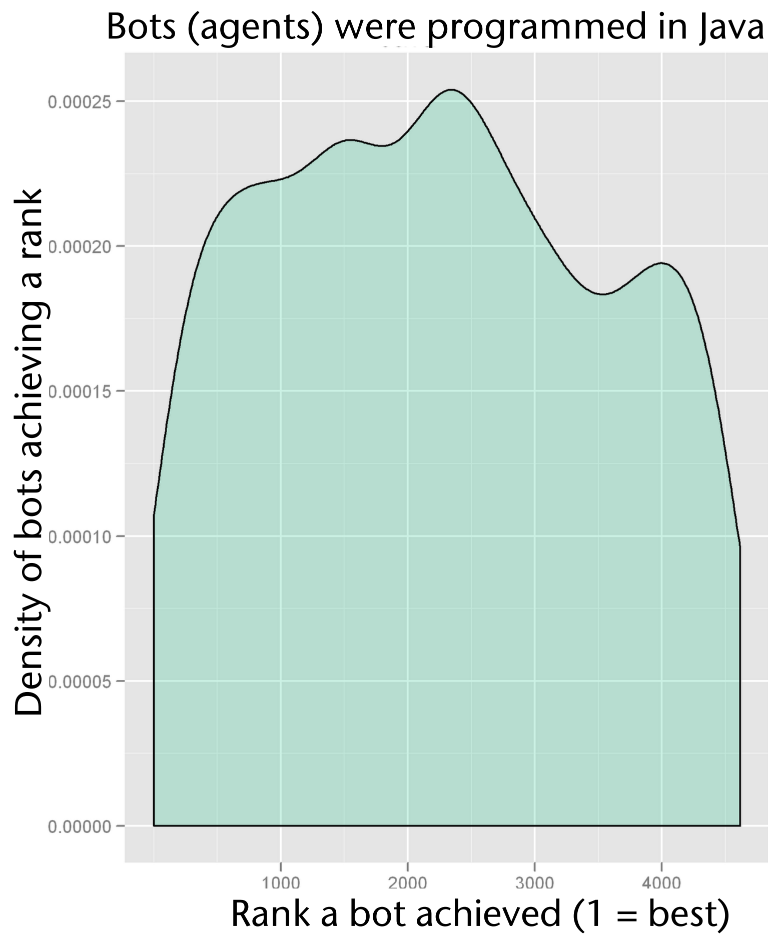
  or (in the discrete world)

$$P[X \leq x] = H(x) = \frac{1}{N} \sum_0^x h(u)$$

Example Histogram (or, rather, PDF)

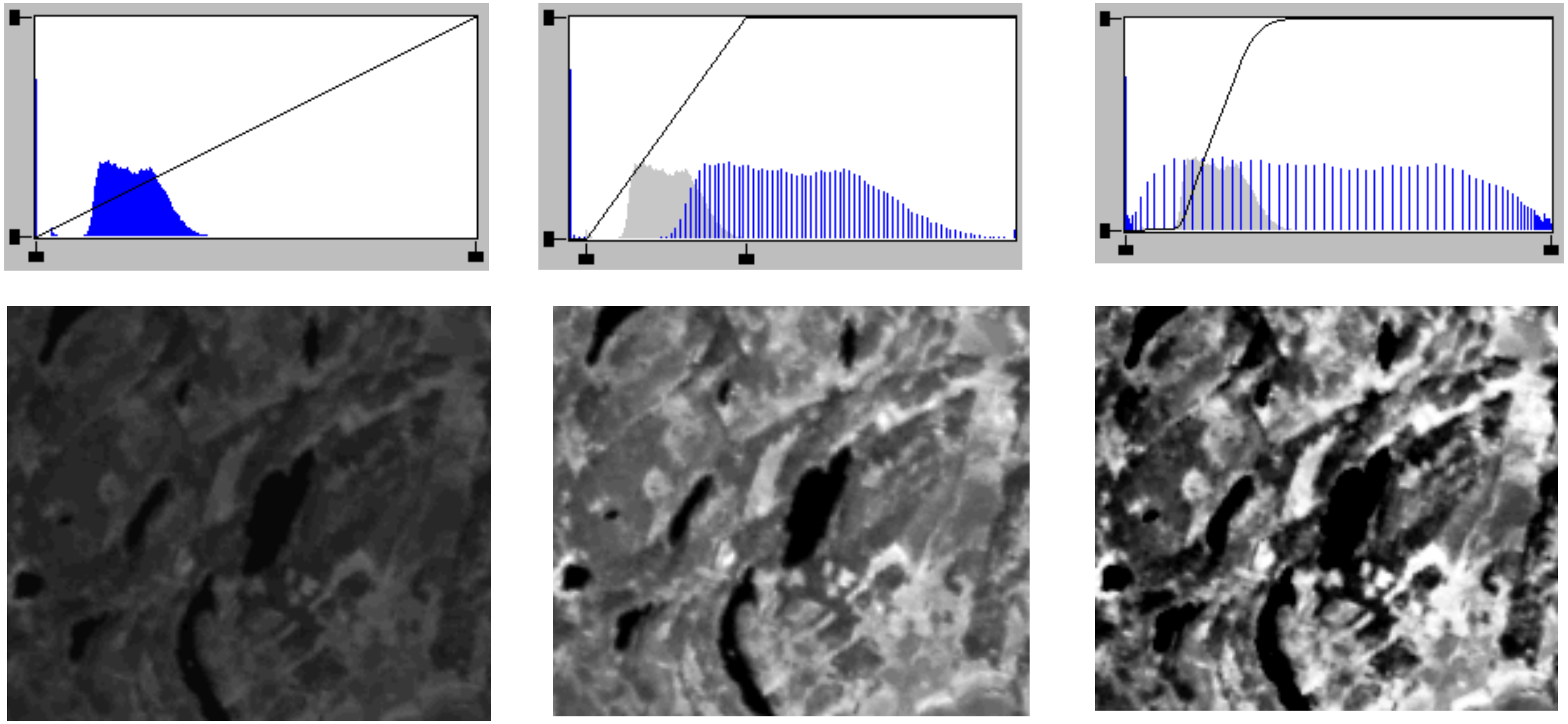
- How did *bots* (= *agents*) or, rather, programmers compare according to programming language in the Google AI challenge 2010:





Can We Do Better Than Histogram Stretching?

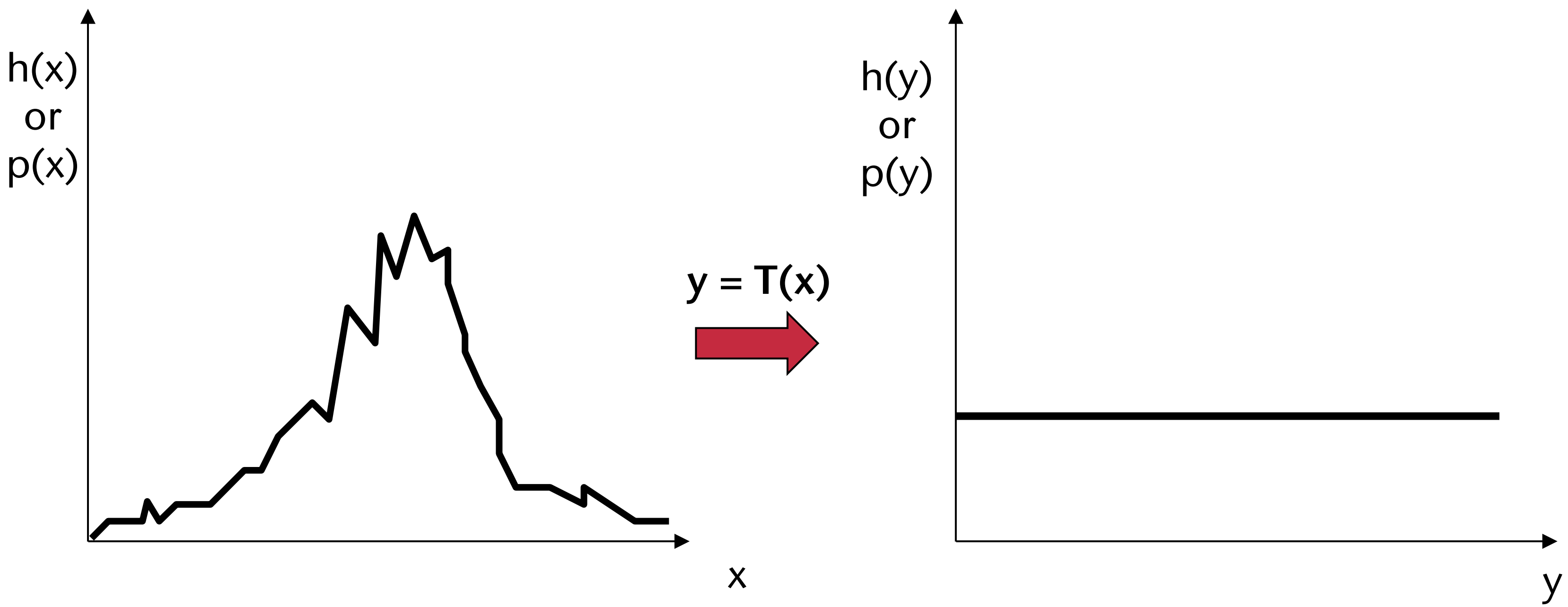
- Example with different transfer function:



- How can we find algorithmically the optimal transfer function?

Histogram Equalization

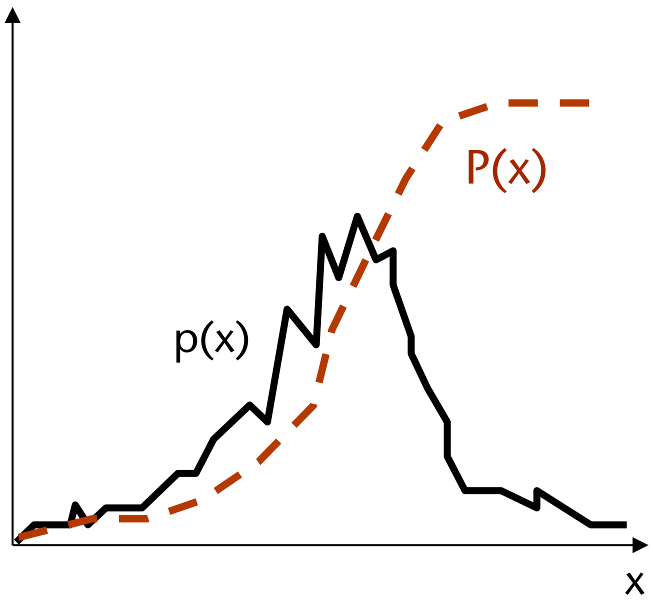
- Given: a random variable X with certain PDF p_X
- Wanted: function T such that the random variable $Y = T(X)$ has a uniformly distributed PDF $p_Y \equiv \text{const}$
- This transformation is called histogram equalization

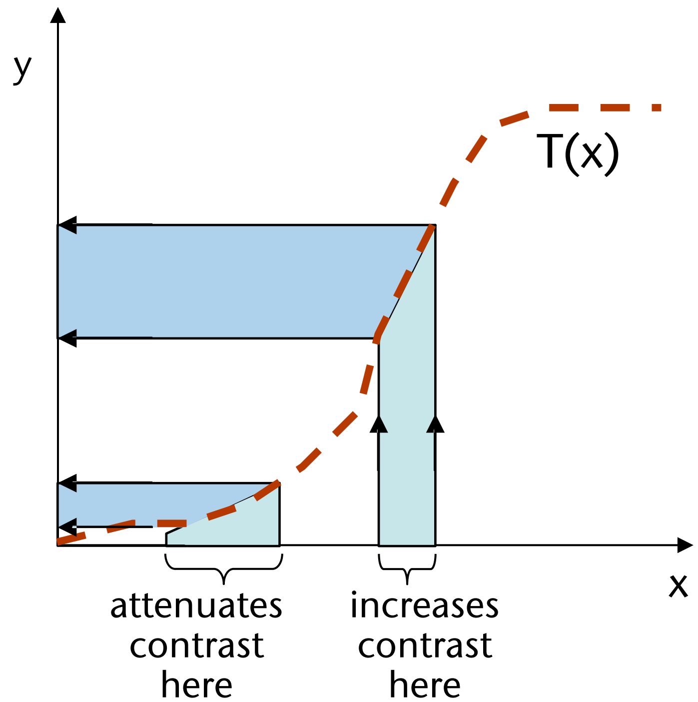


- Conjecture: the transfer function

$$y = P(x) = \int_0^x p(u)\,du$$

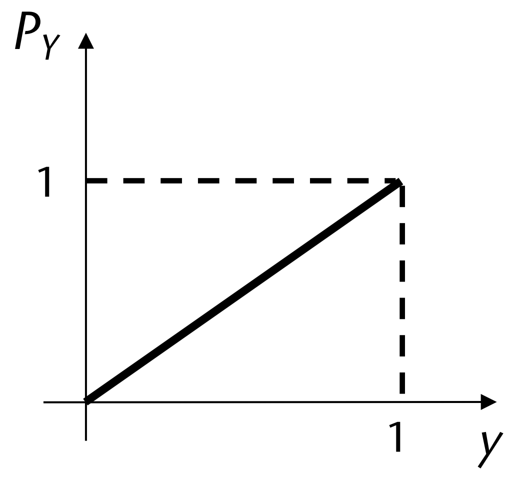
performs exactly this histogram equalization





1. Version of a Proof

- To prove: $P_Y(y) = y$
  - I.e., the image after the transformation by the transfer function has a flat histogram



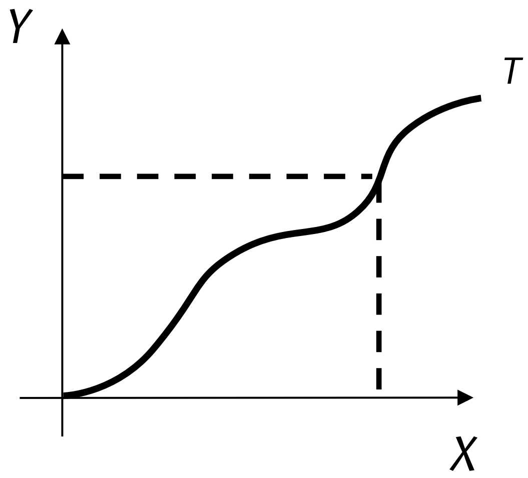
- Proof by inserting:

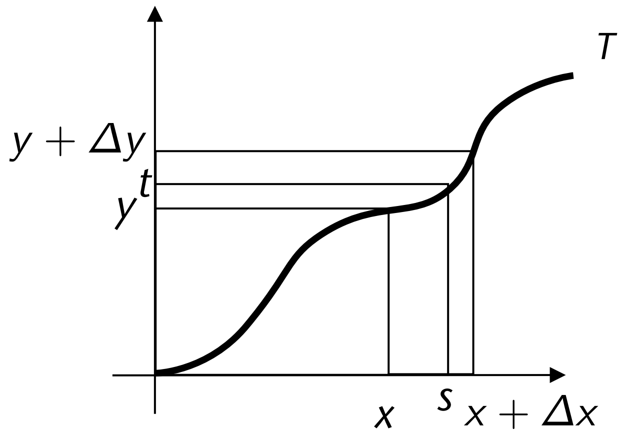
$$
\begin{aligned}
P_Y(y) &= P[Y \leq y] \\
&= P[T(X) \leq y] \\
&= P[P_X(x) \leq y] \\
&= P[x \leq P_X^{-1}(y)] \\
&= P_X(P_X^{-1}(y)) \\
&= y
\end{aligned}
$$

2. Version of a Proof

- Let X be a continuous random variable
- Let $Y = T(X)$ (so Y is a continuous RV, too)
- Let T be $\mathcal{C}^1$ and monotonically increasing
- Consequently, there exist T' and T^{-1}
- Because T maps all $x \leq s \leq x + \Delta x$ to $y \leq t \leq y + \Delta y$, we have

$$\int_x^{x+\Delta x} p_X(s)ds = \int_y^{y+\Delta y} p_Y(t)dt$$





- So, for small Δx, we have

$$p_Y(y)\Delta y \approx p_X(x)\Delta x \qquad p_Y(y) \approx p_X(x)\frac{\Delta x}{\Delta y}$$

- When $\Delta x \to 0$, then the approx'ion becomes an exact equation:

$$p_Y(y) = \lim_{\Delta x \to 0} p_X(x) \frac{\Delta x}{\Delta y} = p_X(x) \lim_{\Delta x \to 0} \frac{1}{\Delta y / \Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{T(x + \Delta x) - T(x)}{\Delta x} = T'(x)$$

- Combined:

$$p_Y(y) = \frac{p_X(x)}{T'(x)}$$

- Now, inserting $x = T^{-1}(y)$ results in

$$p_Y(y) = \frac{p_X(T^{-1}(y))}{T'(T^{-1}(y))}$$

- Side result: now we know how to convert distribution functions, if a random variable is a function of another random variable.

- Continue with the histogram equalization ...

- Sought is a function *T*, such that

$$p_Y(y) \equiv 1$$

- Inserting our previous results yields

$$\frac{p_X(T^{-1}(y))}{T'(T^{-1}(y))} = 1$$

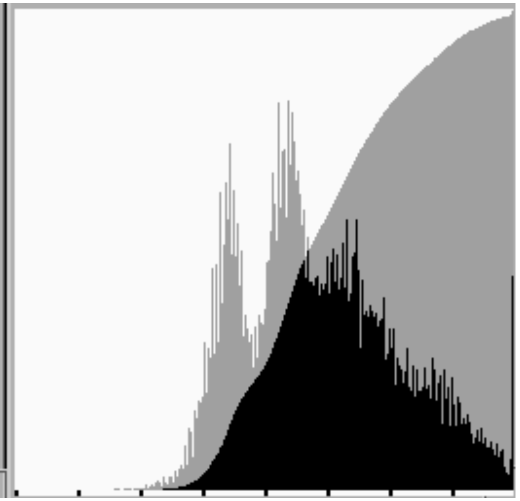
$$T'(T^{-1}(y)) = p_X(T^{-1}(y))$$

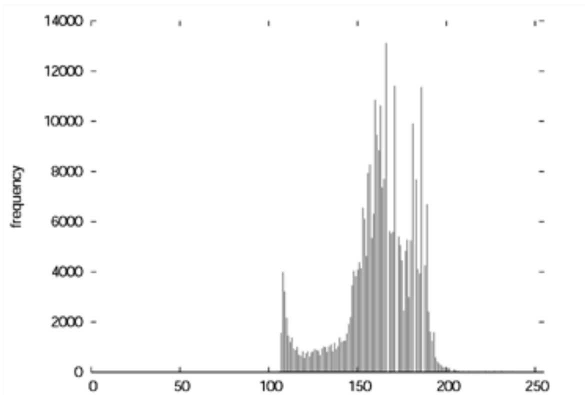
- Inserting $x = T^{-1}(y)$ results in $T'(x) = p_X(x)$
- Sought was *T*, so integration yields:

$$T(x) = \int_0^x T'(u)du = P_X(x)$$

Examples

Orig. Image	Histogram	Result

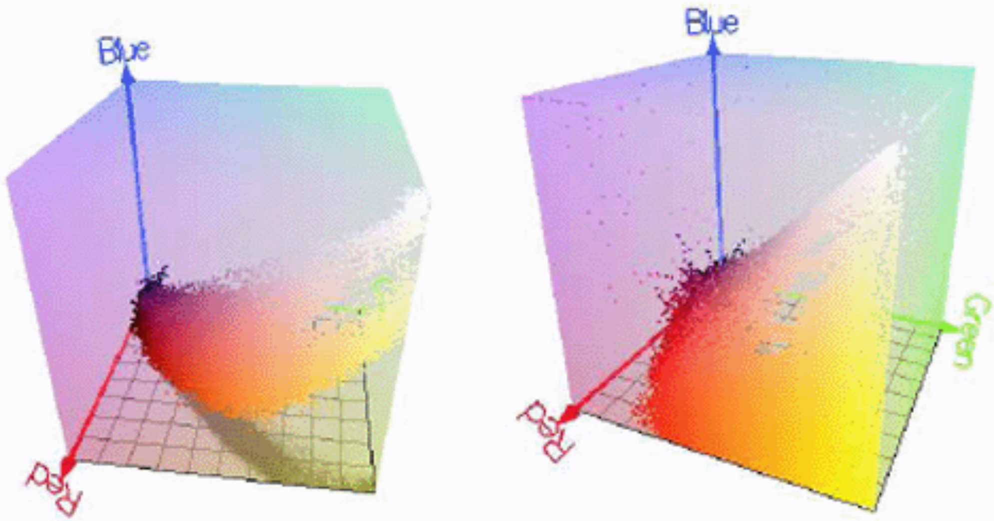




Equalization in HSV

Original Image

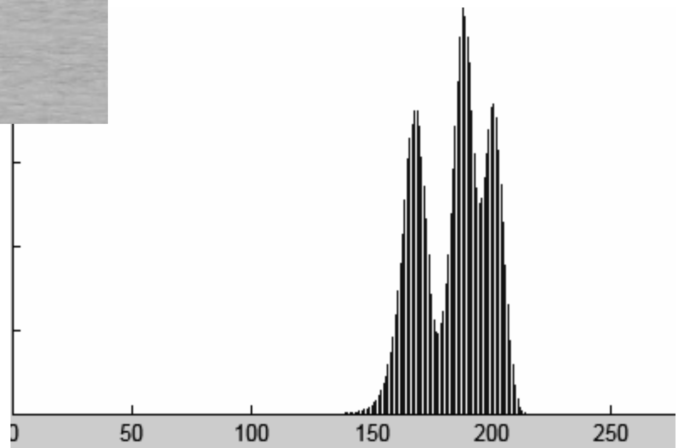
Equalized Image
a.k.a. probability smoothing)



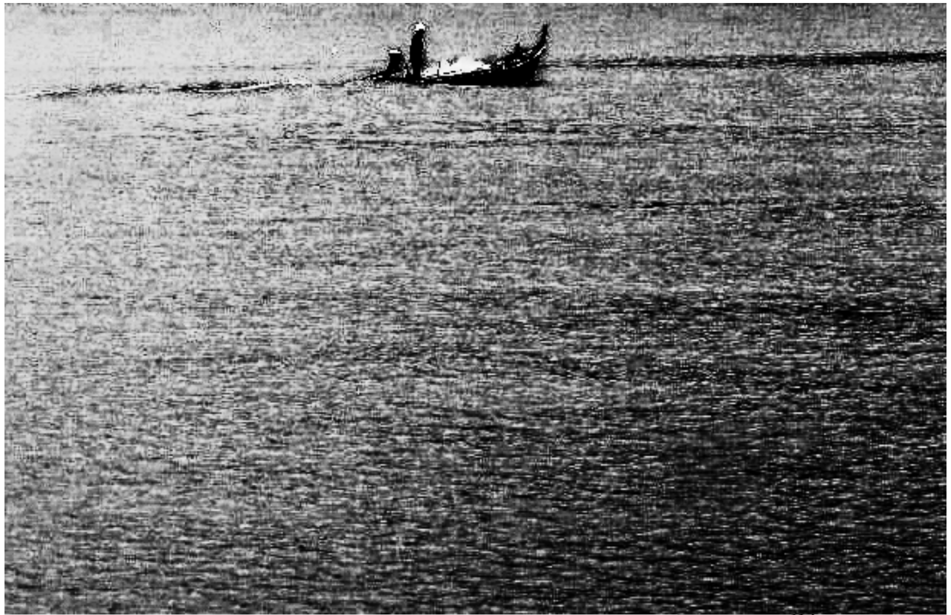
Equalization in RGB

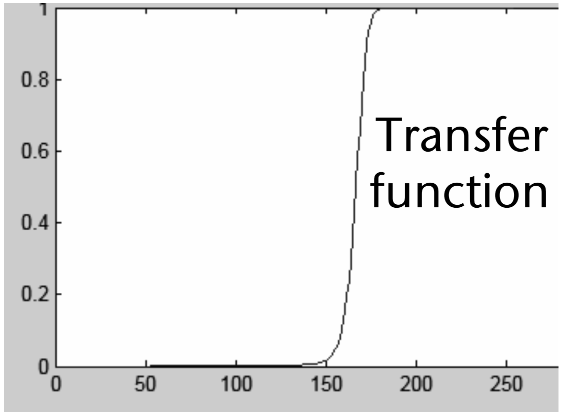
A Problem of Histogram Equalization

- Problematic case: a very narrow histogram of the input image

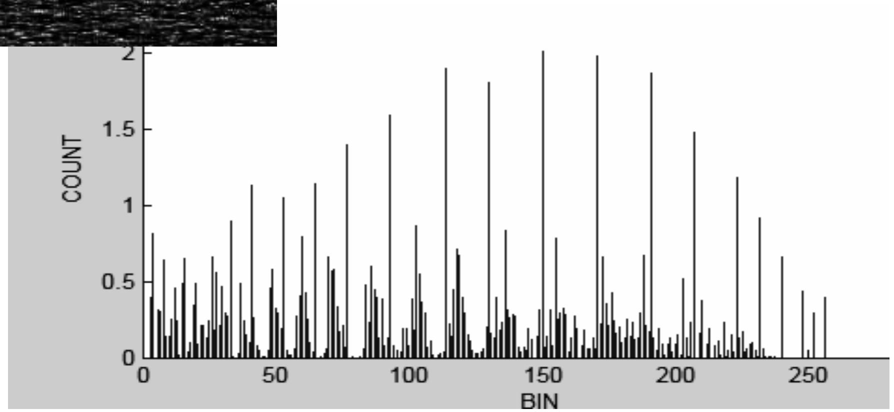


- Result: unwanted contrast



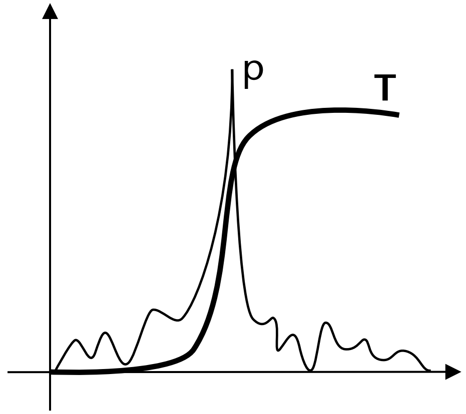


Resulting histogram



Tone Reproduction by Ward et al. [1997]

- Problem of histogram equalization:
  - Very steep sections of the transfer function T can produce visible noise
- Idea: limit the slope of T
- Algorithm:



1. Determine the histogram h
   - Reminder: $h \approx p = T'$
2. Clamp too large bins to a value $\alpha \cdot \frac{N}{B}$, where $\alpha \approx 0.5 \ldots 1.5$, N = number of pixels, B = number of bins
3. Let $N' = \sum_{i=0}^{L-1} h(x_i)$
4. Use this to perform equalization and repeat a few times

Excursion: The Weber-Fechner Law [~1850]

- By experiment, we find:
  - The just noticeable difference (JND) of a stimulus (e.g., weight) depends on the *level* of the stimulus (differential threshold of noticeability)
  - The ratio of the JND over the level of the stimulus is constant (depending on the kind of stimulus)
- The mathematical formulation of these findings:
  - Let S be the level of the stimulus, and let ΔS be the JND at this level
  - Now, Weber's law says:

$$\frac{\Delta S}{S} = \text{const}$$

- The Weber-Fechner law:

  Let E be the level of the perceived sensation of S (e.g., perceived weight), and let ΔE be the JND of E.

  Then we have

$$\Delta E = k \frac{\Delta S}{S} \quad \Rightarrow \quad \frac{dE}{dS} = k \frac{1}{S}$$

- Integration results in:

$$E = k \cdot \ln S + c$$

  - Here, c is a constant that describes the minimum stimulus S_0, with which just a sensation $E \approx 0$ is created (threshold stimulus):

$$c = -k \cdot \ln S_0$$

- Combined:

$$E = k \cdot \ln \frac{S}{S_0}$$

Excursion²: The Stevens Power Function

- Another plausible assumption seems (IMHO) the following:

$$\frac{\Delta E}{E} = k \frac{\Delta S}{S}$$

- Transformation results in:

$$\frac{1}{E}\Delta E - k\frac{1}{S}\Delta S = 0 \quad \Rightarrow$$

$$\frac{1}{E}\mathrm{d}E - k\frac{1}{S}\mathrm{d}S = 0 \quad \Rightarrow \quad \ln E - k \ln S = c \quad \Rightarrow$$

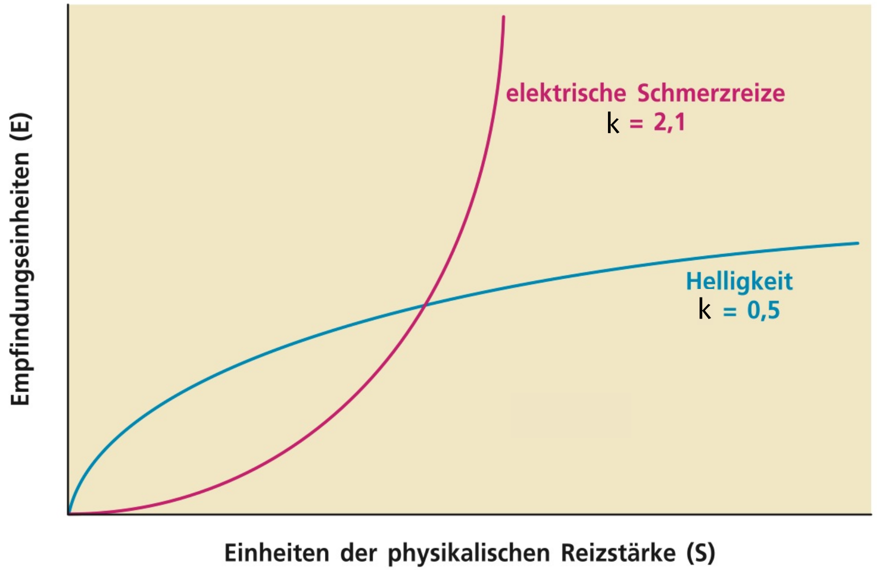
$$\ln \frac{E}{S^k} = c \quad \Rightarrow \quad \frac{E}{S^k} = e^c = c' \quad \Rightarrow$$

- Finally results in Stevens' power law:

$$E = cS^k$$

where E = sensation strength ("perceived weight"), S = stimulus (a physical value), c and k = constants, which depend on the sense organ

- For many stimuli, $k < 1$
(for brightness k ≈ 0.5,
for volume k ≈ 0.6)
- For some stimuli, $k > 1$
(for temperature k ≈ 1-1.6,
for electric shock k ≈ 2-3)



Notes on the Laws

- The Weber-Fechner law describes (apparently) better the perception of stimuli in the middle range, the Stevens power law better in the lower and upper range
- Research on the two laws is still in full swing
- There are early indications that neural networks and cellular automata also show this behavior, if sensory perception (excitation + transport) is simulated with them!